Non-Trivial Spin Obliquity Attractors in Multi-Planetary Super-Earth Systems

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2025/12/19

Based on
Su & Lai (2020)
Su & Lai (2022a,b)
DL's talk slides
Guo & Lai (~2026)

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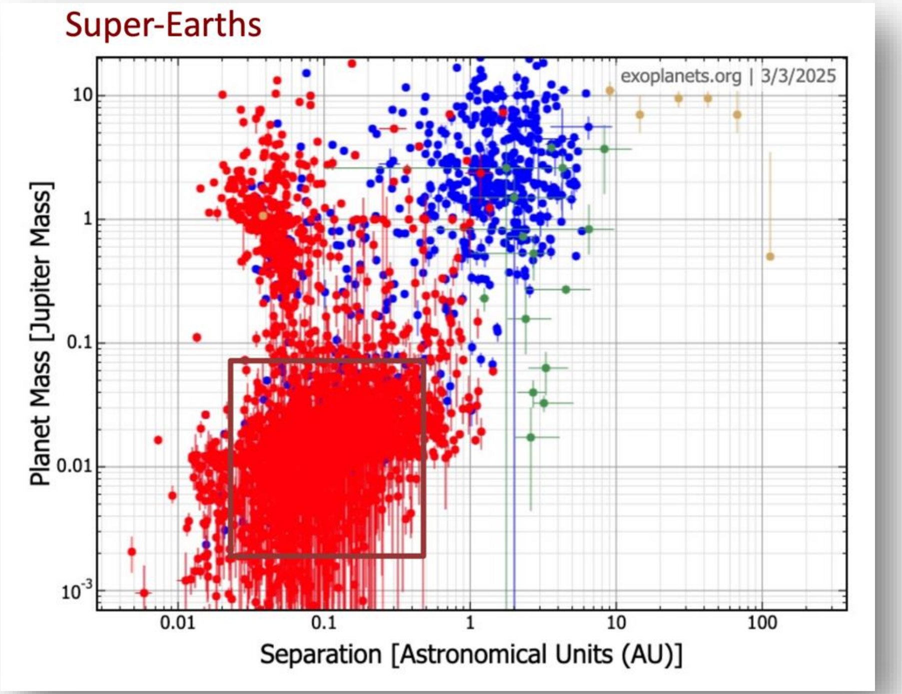
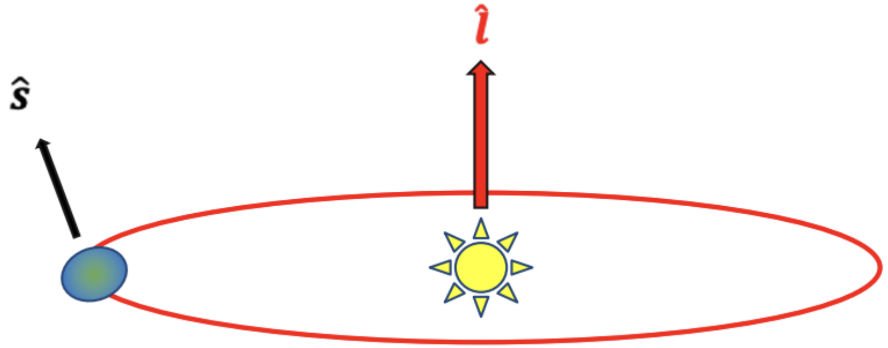
Background: super earth obliquity



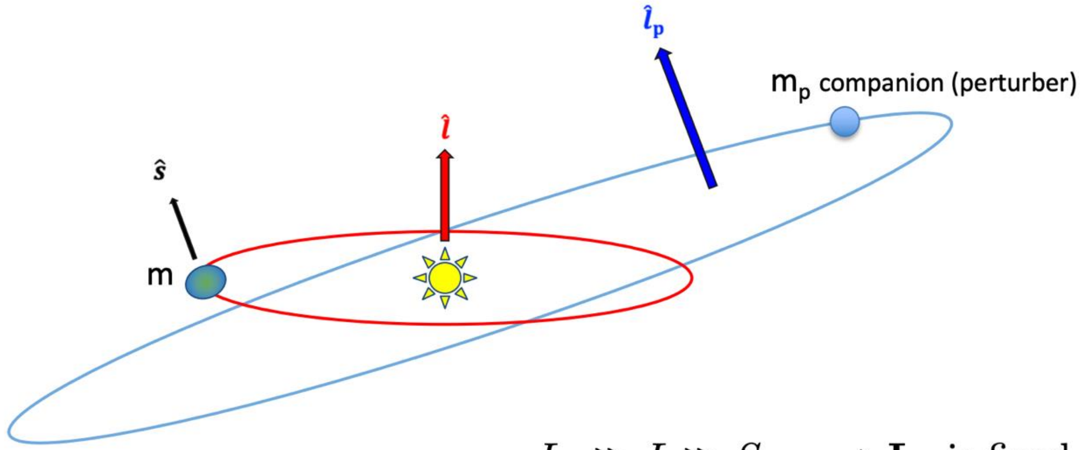
Super-earths in multi-planet systems tend to have *significant obliquities*



Two stages of SE obliquity evolution:

- Giant collision (merger, collision, scattering) generating large obliquities
- Long-term evolution
    - Tidal dissipation (for now, alignment torque)
    - Perturbation (from star & outer companion)

Model Baseline: (conservative) Colombo's top



$L_p \gg L \gg S \quad \Longrightarrow \mathbf{L}_p$ is fixed

Case with 1 perturber is *integrable* (not chaotic, analytically solvable)

EoM in stellar frame:

$$\frac{\mathrm{d}\hat{\mathbf{l}}}{\mathrm{d}t} = \omega_{\mathrm{lp}} \left(\hat{\mathbf{l}} \cdot \hat{\mathbf{l}}_{\mathrm{p}}\right) \left(\hat{\mathbf{l}} \times \hat{\mathbf{l}}_{\mathrm{p}}\right) \equiv -g \left(\hat{\mathbf{l}}_{\mathrm{p}} \times \hat{\mathbf{l}}\right), \quad \text{(Orbit precession)}$$

$$\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t} = \omega_{\mathrm{sl}} \left(\hat{\mathbf{s}} \cdot \hat{\mathbf{l}}\right) \left(\hat{\mathbf{s}} \times \hat{\mathbf{l}}\right) \equiv -\alpha \left(\hat{\mathbf{s}} \cdot \hat{\mathbf{l}}\right) \left(\hat{\mathbf{l}} \times \hat{\mathbf{s}}\right), \quad \text{(Spin precession)}$$

Changing to L-frame:

$$\boxed{\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \alpha \left(\hat{\mathbf{s}} \cdot \hat{\mathbf{l}}\right) \left(\hat{\mathbf{s}} \times \hat{\mathbf{l}}\right) + g \left(\hat{\mathbf{l}}_{\mathrm{p}} \times \hat{\mathbf{s}}\right)}$$

Hamiltonian formalism:

- **# DoF: only 1**
- $(p, q) \equiv (\cos\theta_{\mathrm{sl}},\ \phi_{\mathrm{sl}})$
- Solution/phase flow along H_0 contour

Non-dimensionalized:

$$\mathcal{H}_0 = \mathcal{H}_0(p, q) = -\frac{1}{2}p^2 + g\left(p\cos I - \sin I \cos q \sqrt{1 - p^2}\right)$$

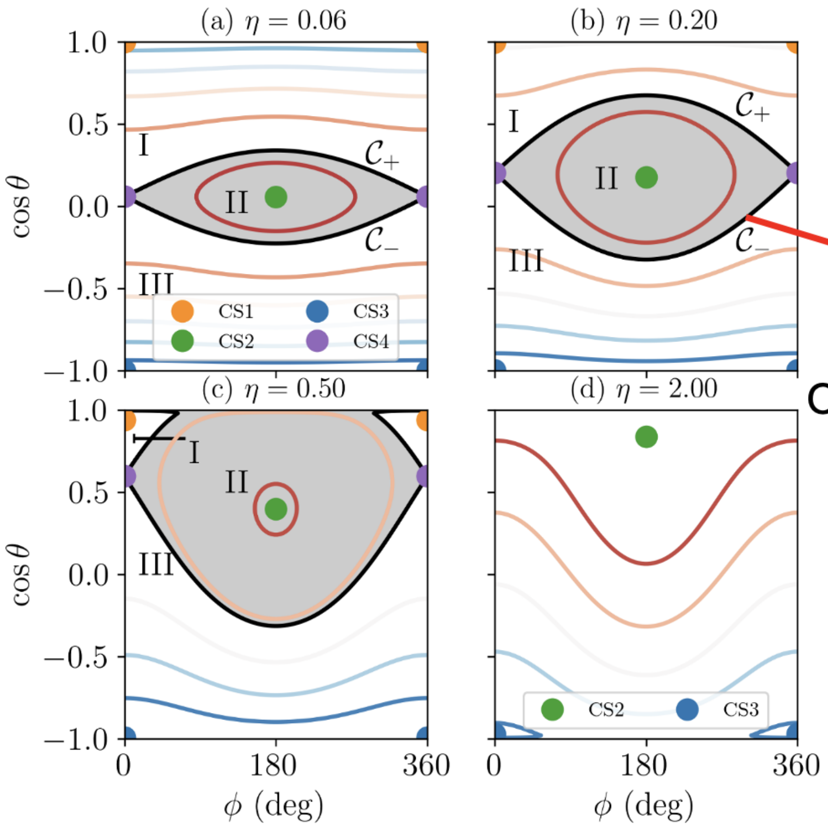
(A purely geometric problem...)

Model Baseline: (conservative) Colombo's top

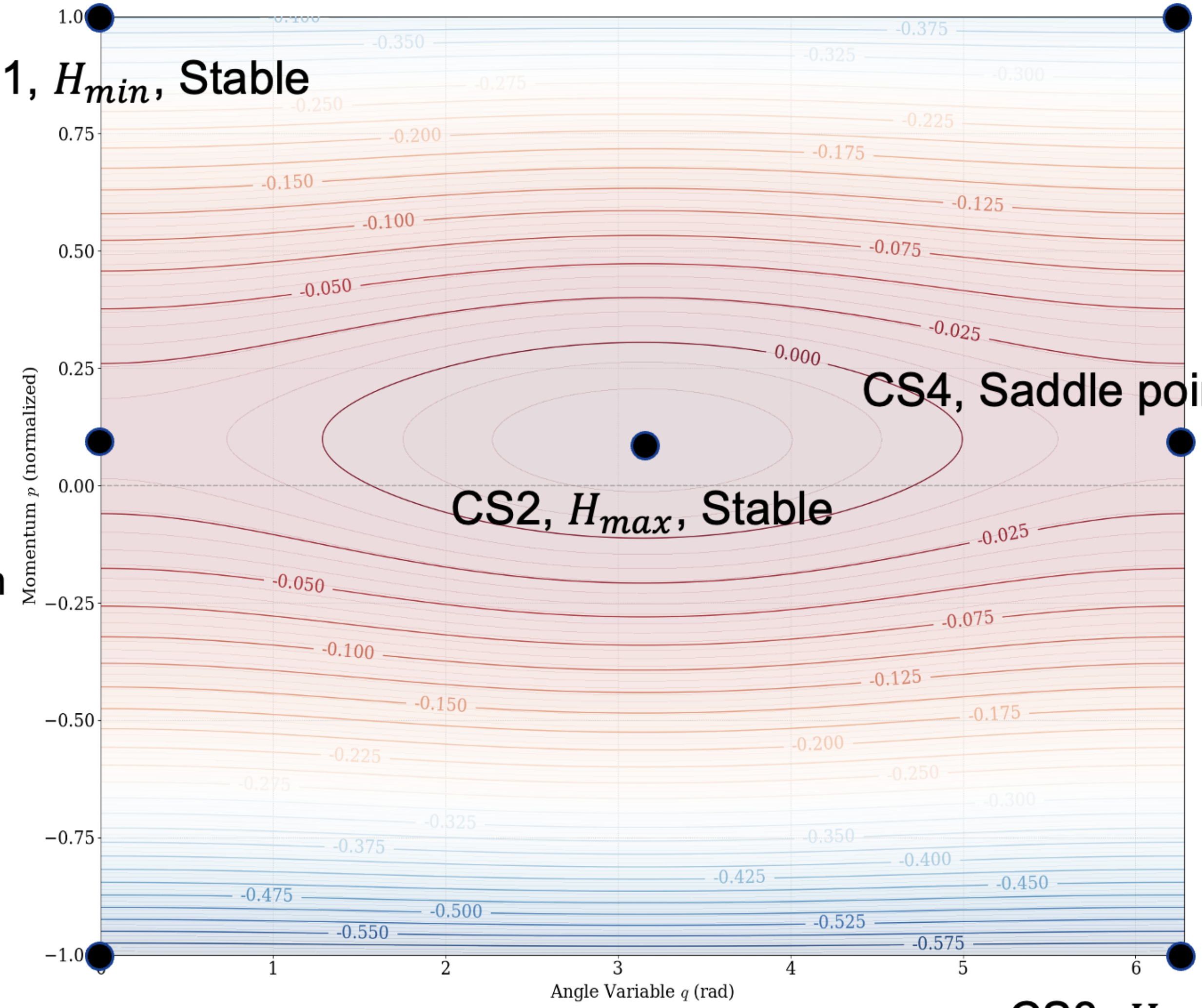
Without dissipation:

$$\mathcal{H}_0 = \mathcal{H}_0(p, q) = -\frac{1}{2}p^2 + g\left(p\cos I - \sin I \cos q\sqrt{1-p^2}\right)$$

Su&Lai, 2022a



Separatrix
In: libration
Out: circulation



Case for smaller *g* is of interest

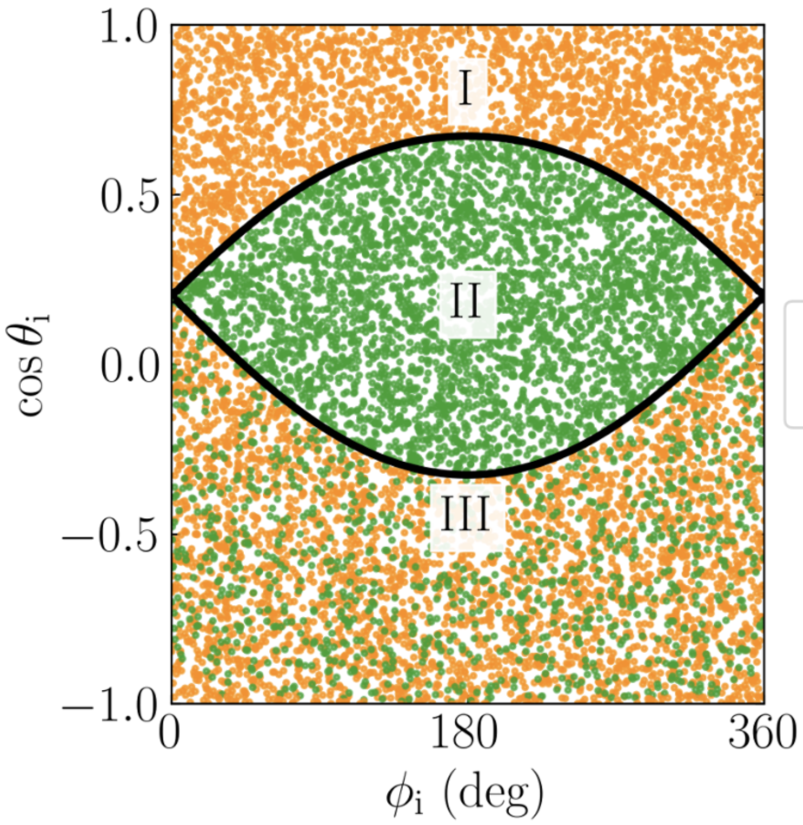
Model Baseline: (tidal) Colombo's top

Adding tidal alignment torque, system evolves (adiabatically) ONLY into CS1 or CS2, depending on IC

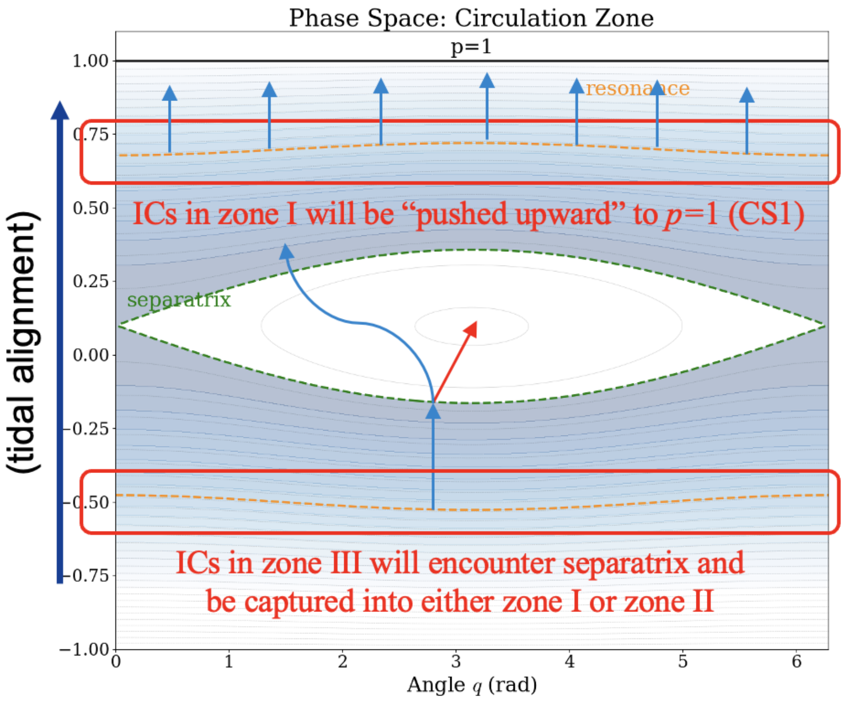
Tidal dissipation: (weak/adiabatic)

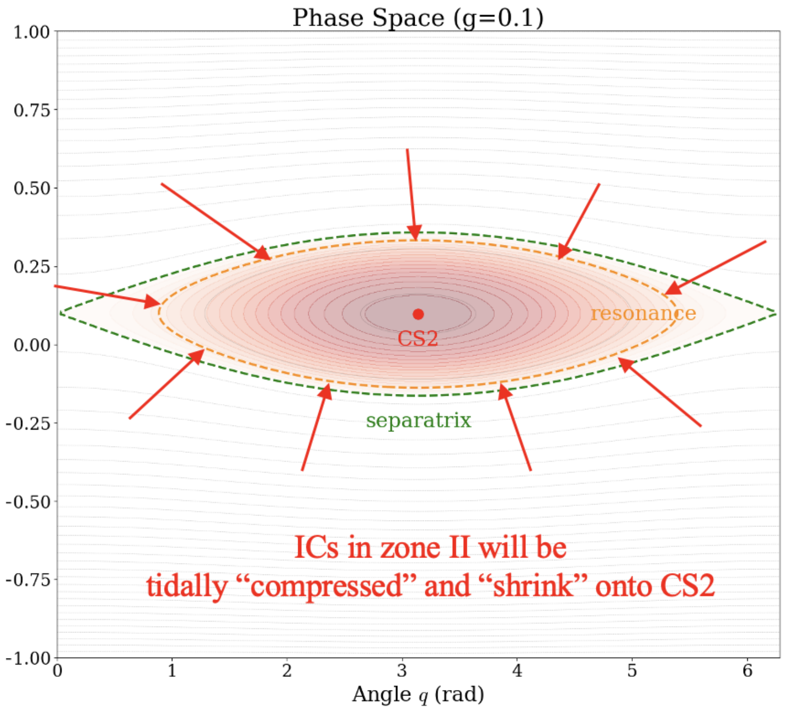
Newton: $\left(\frac{d\hat{s}}{dt}\right)_{\text{tide}} = t_{\text{al}}^{-1}\,\hat{s} \times \left(\hat{l} \times \hat{s}\right)$

Hamiltonian: $\dot{p}|_{\text{tide}} = t_{\text{al}}^{-1}\,(1-p^2) \geq 0$

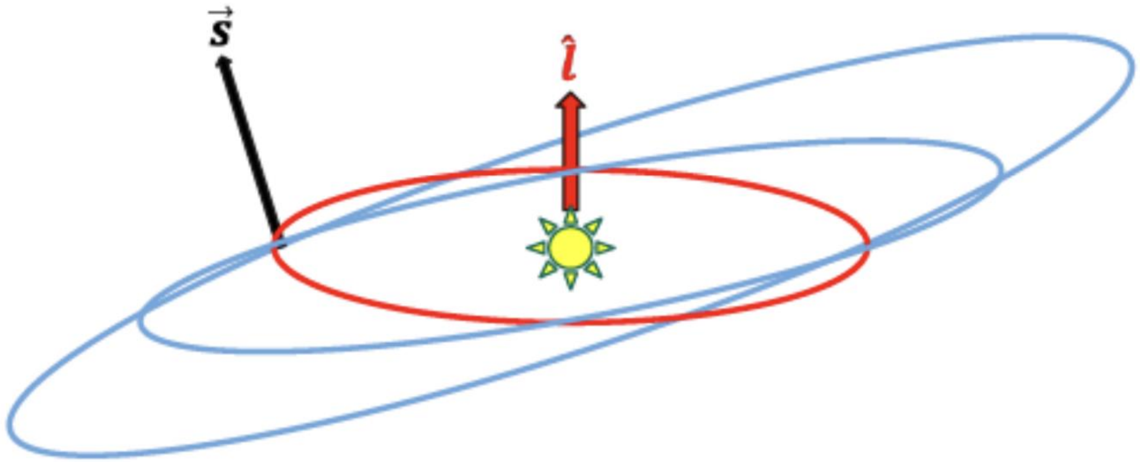


Su&Lai, 2022a





Model Extension: adding 2nd companion...



EoM in rotating/precessing frame:

$$\dot{p} = \sqrt{1-p^2}(\dot{I}\cos q + I\dot{\Omega}\sin q) + \frac{1}{t_{al}}(1-p^2)$$
$$\dot{q} = -\alpha p - \dot{\Omega}\sqrt{1-I^2} + (\dot{I}\sin q - I\dot{\Omega}\cos q)\frac{p}{\sqrt{1-p^2}},$$

Spin evolution: $\frac{d\hat{\mathbf{S}}}{dt} = \alpha\left(\hat{\mathbf{S}}\cdot\hat{\mathbf{L}}\right)\left(\hat{\mathbf{S}}\times\hat{\mathbf{L}}\right)$

Orbital AM: $\mathcal{I} \equiv I\exp(i\Omega) = I_1\exp(-ig_1t) + I_2\exp(-ig_2t),$

$$\hat{\mathbf{L}} = \mathrm{Re}(\mathcal{I})\,\hat{\mathbf{X}} + \mathrm{Im}(\mathcal{I})\,\hat{\mathbf{Y}} + \sqrt{1-|\mathcal{I}|^2}\,\hat{\mathbf{Z}}.$$

Spin precesses around precessing L of 2 frequencies

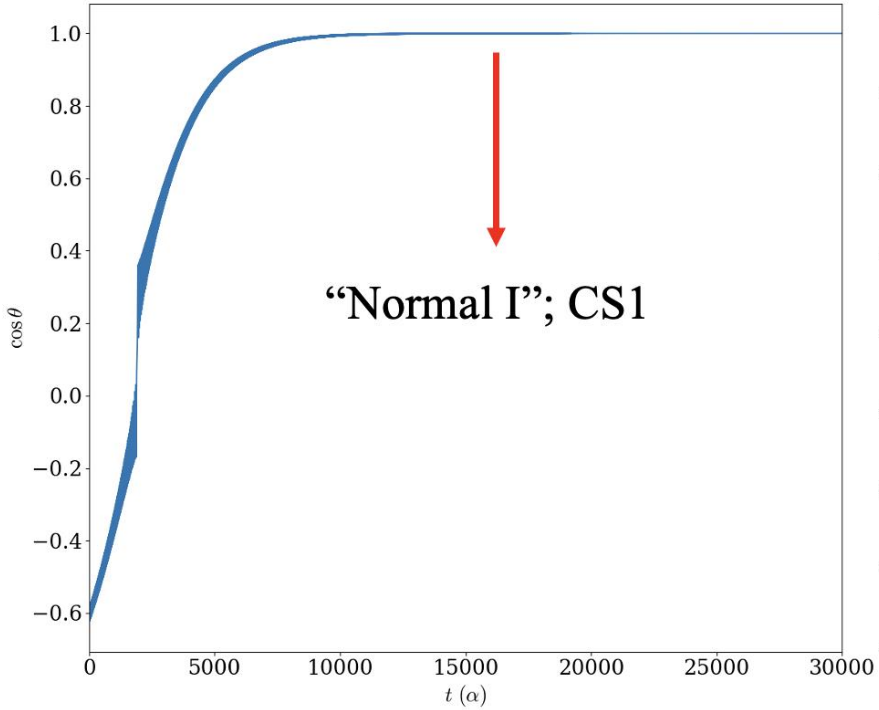
Su&Lai, 2022b

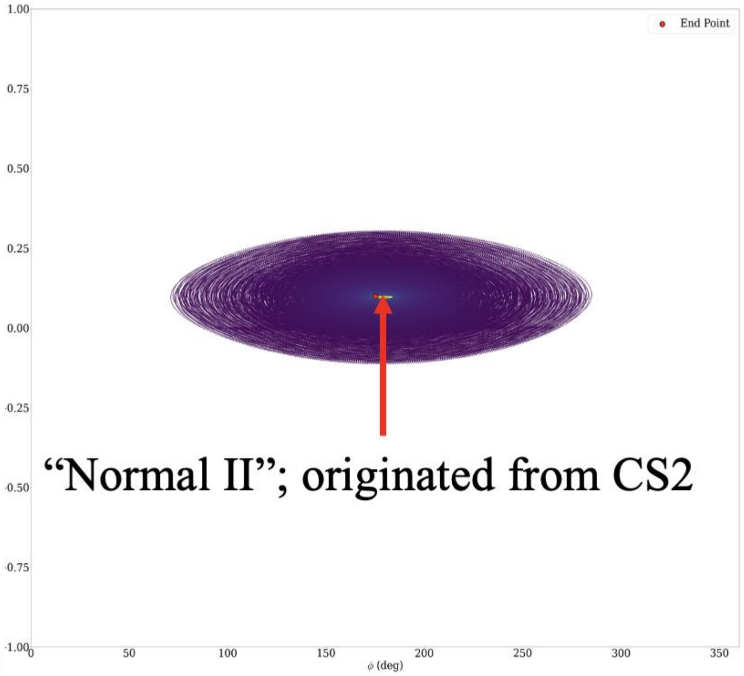
Seems nothing new, but...

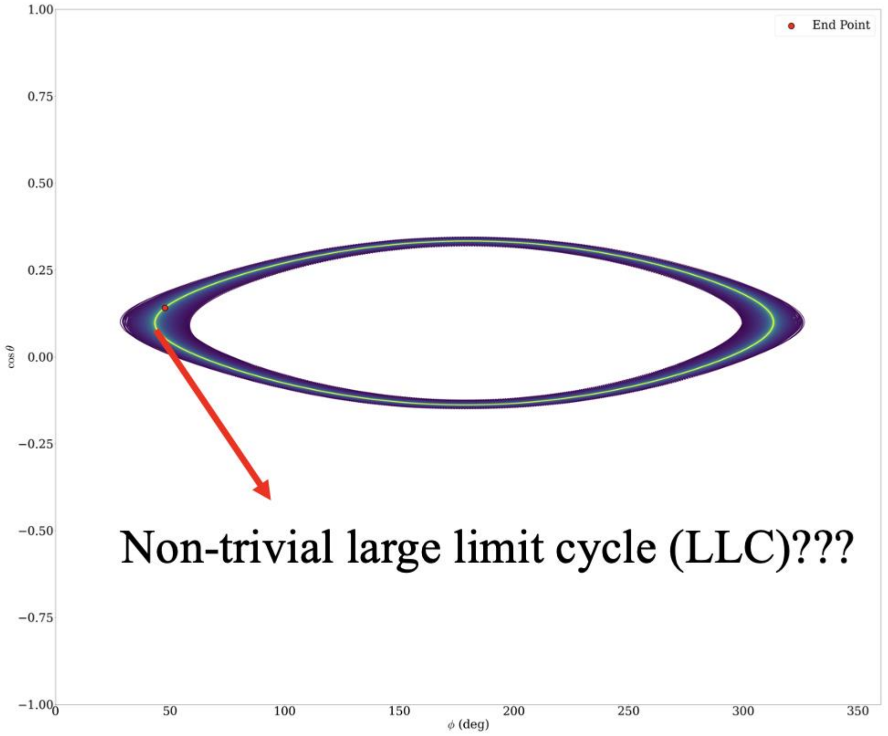
Steady States: normal & non-trivial

For I2<<I1 as perturbation...

$I_1 = 10°, I_2 = 1°; g_1 = 0.1, g_2 = 0.01$







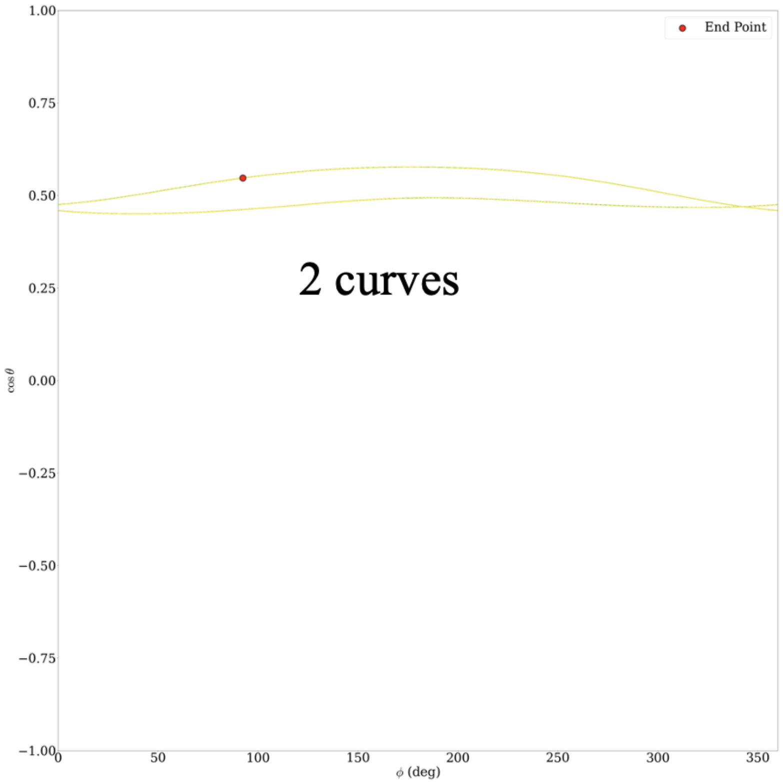
Extraordinary Libration

Non-Trivial Steady States

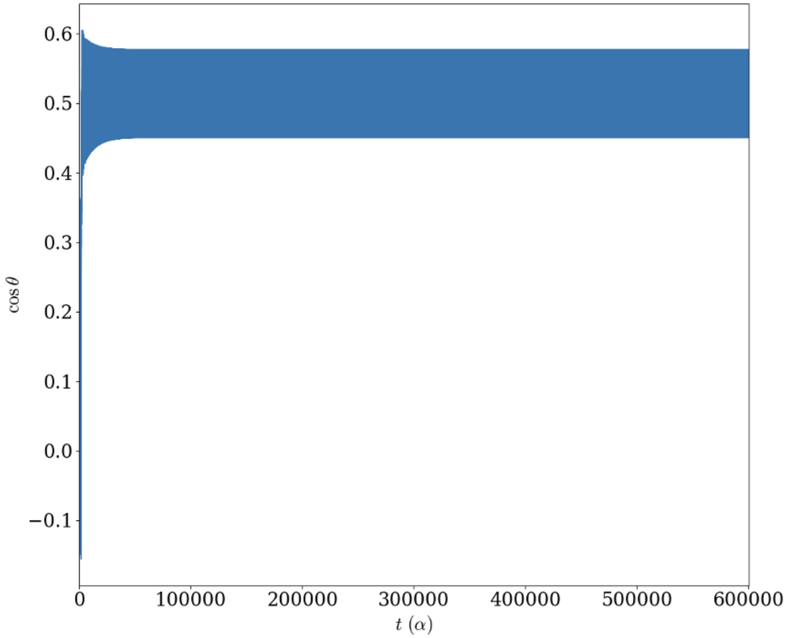
Increasing g2 yields non-trivial circulation steady states

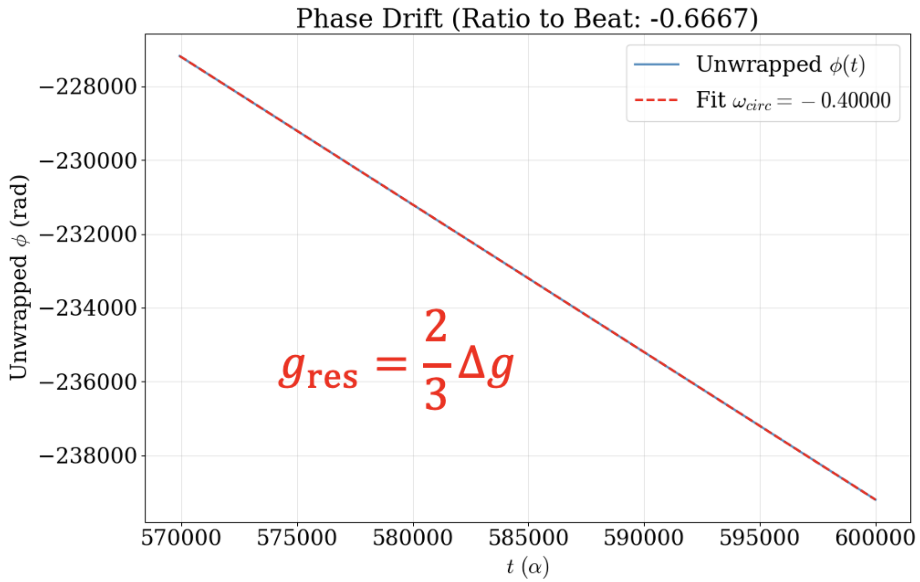
$I_1 = 10°, I_2 = 1°; g_1 = 0.1, g_2 = 0.7, \Delta g = 0.6$

circulation frequency: $\langle \dot{\phi}_{\mathrm{sl}} \rangle = g_{\mathrm{res}} \neq 0$!!!



Stable phase trajectory





Fitting circulating frequency

Non-Trivial Steady States

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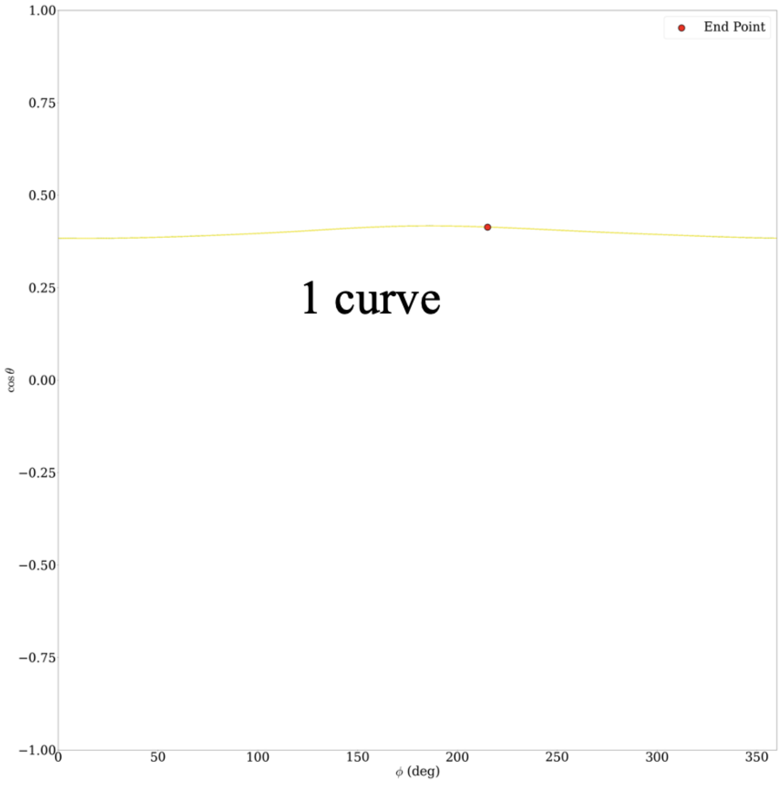
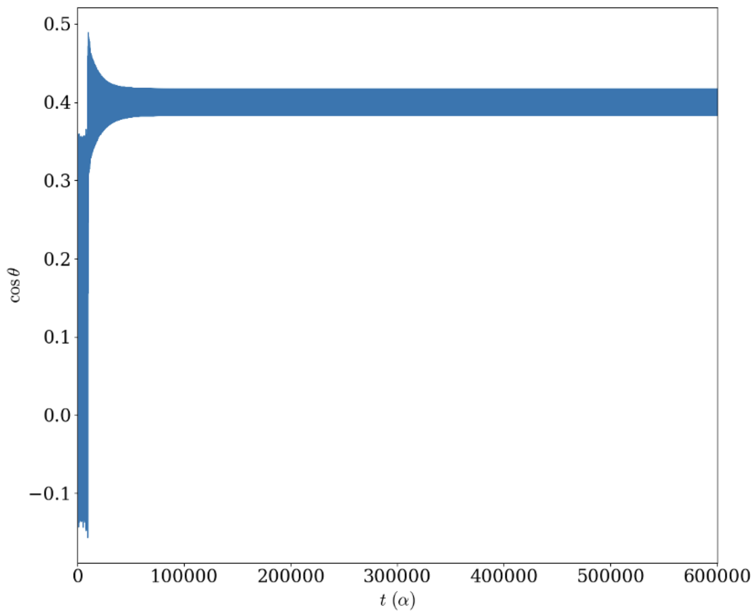
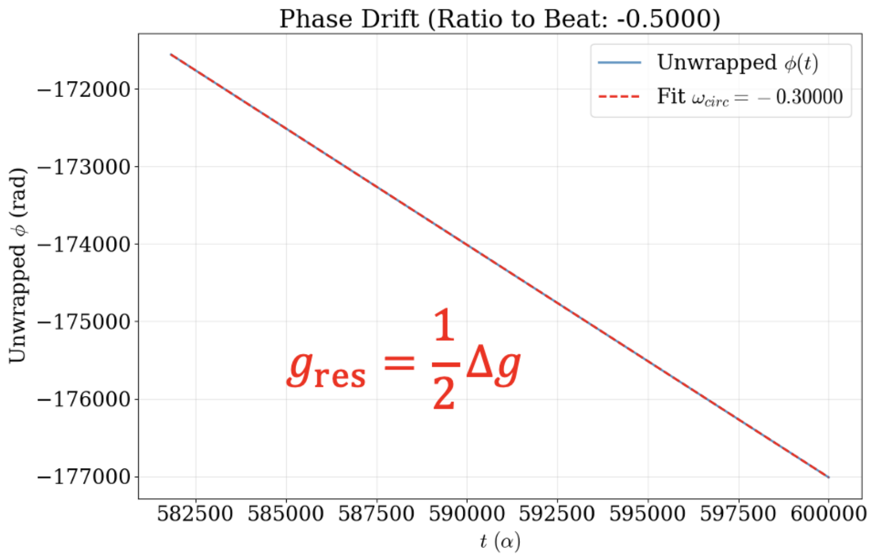


Stable phase trajectory





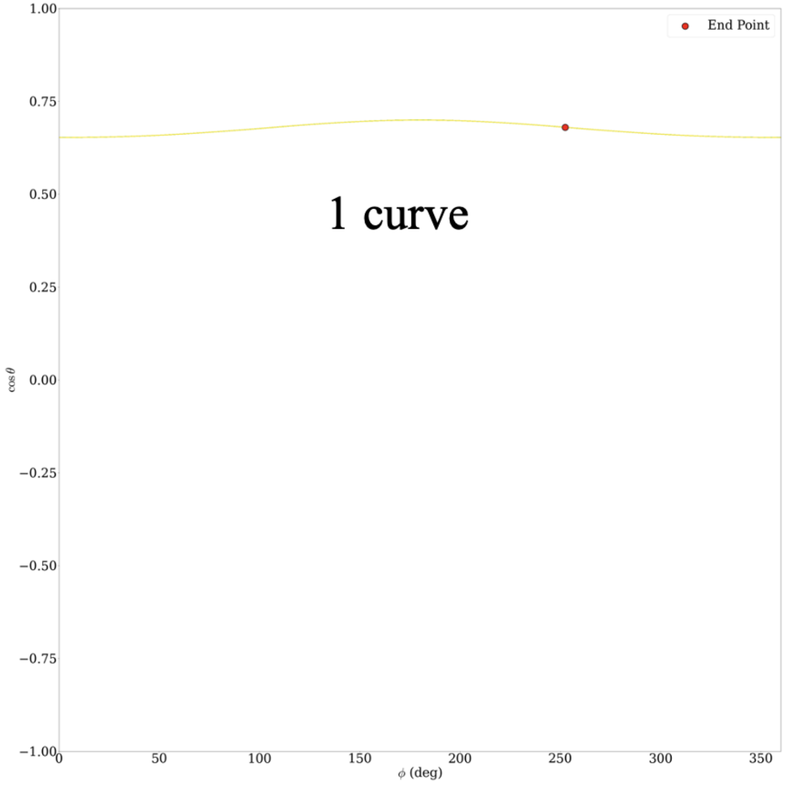
Fitting circulating frequency

Non-Trivial Steady States

Increasing g2 yields non-trivial circulation steady states

$I_1 = 10°, I_2 = 2°; g_1 = 0.1, g_2 = 0.7, \Delta g = 0.6$

circulation frequency: $\langle \dot{\phi}_{sl} \rangle = g_{res} \neq 0$!!!



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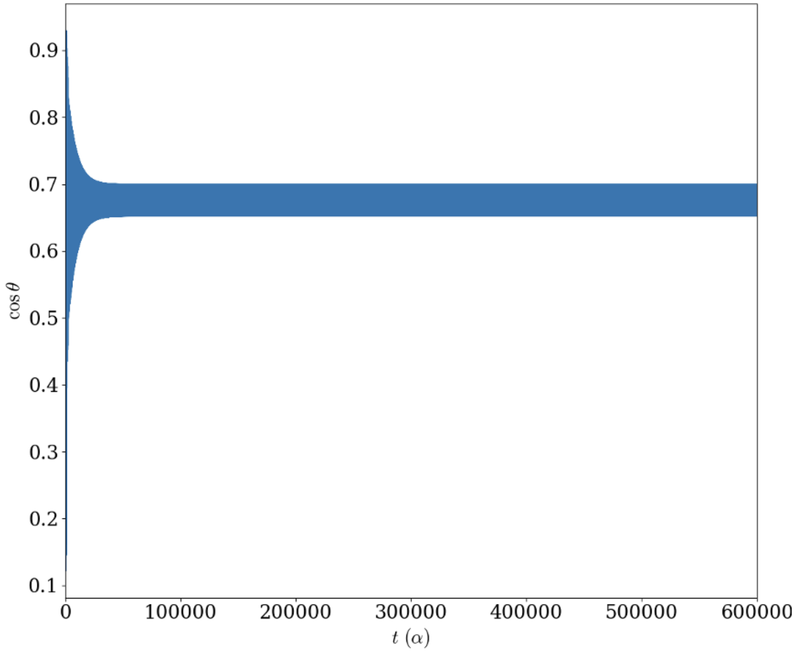
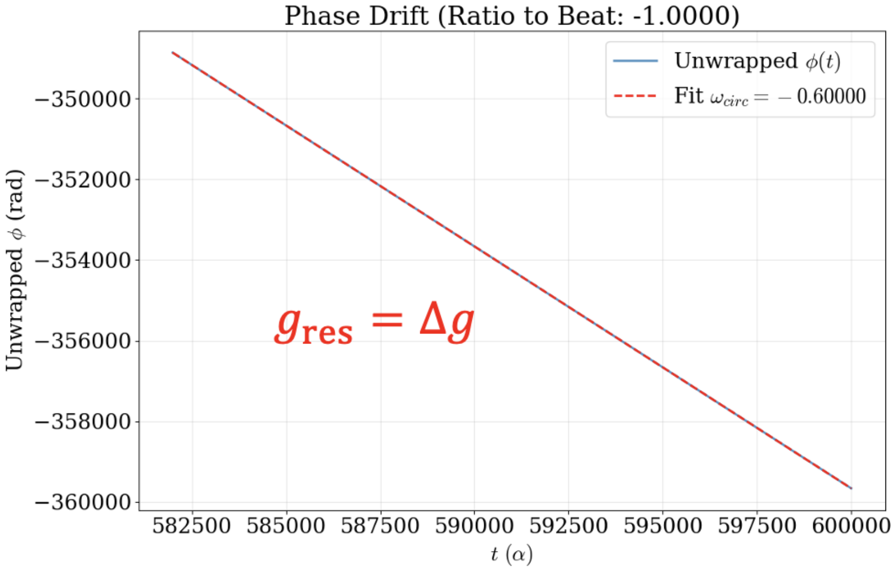
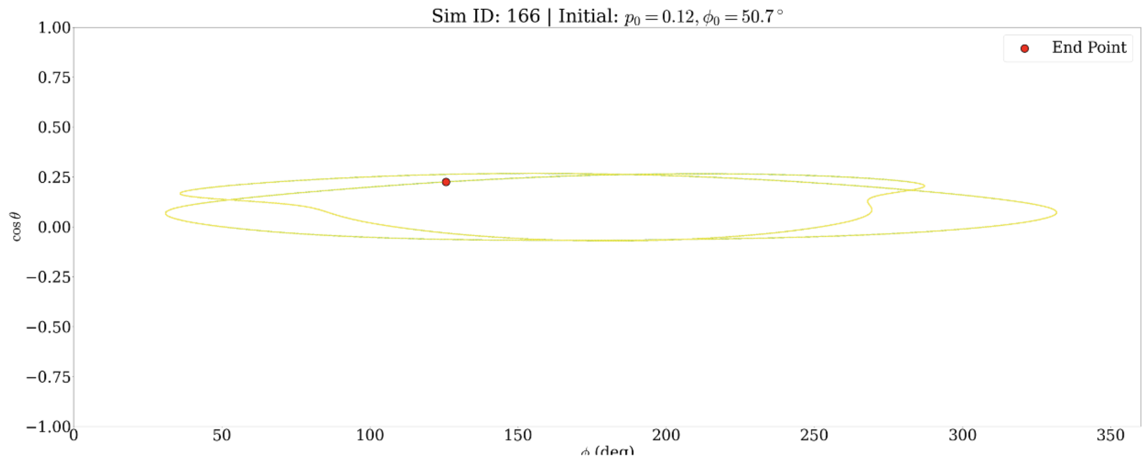


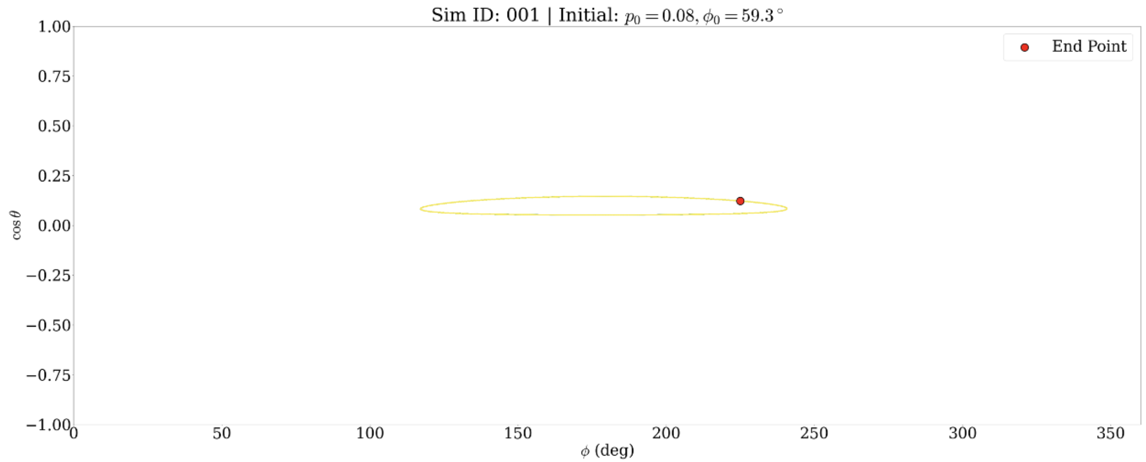


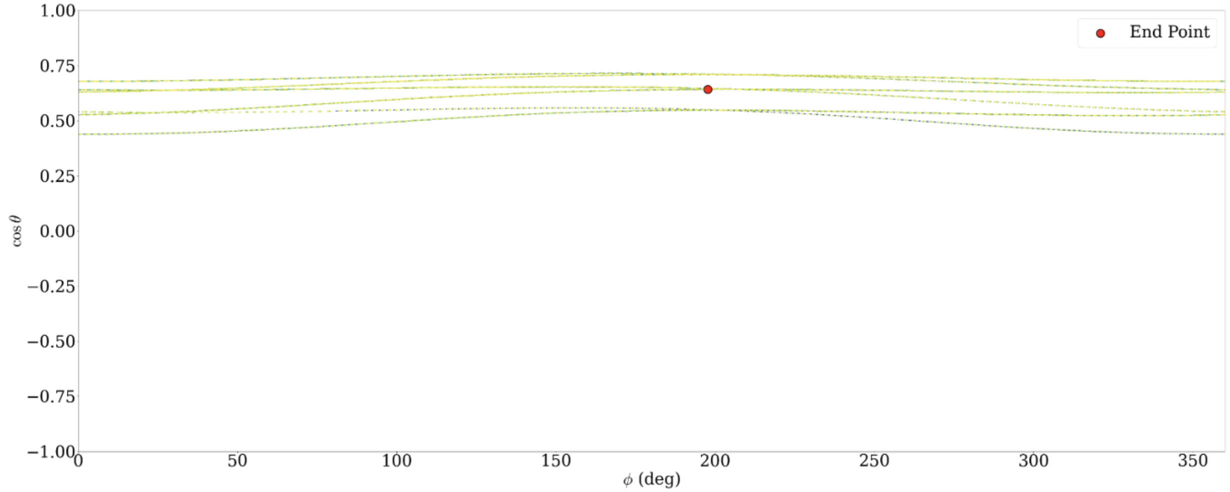
Fitting circulating frequency

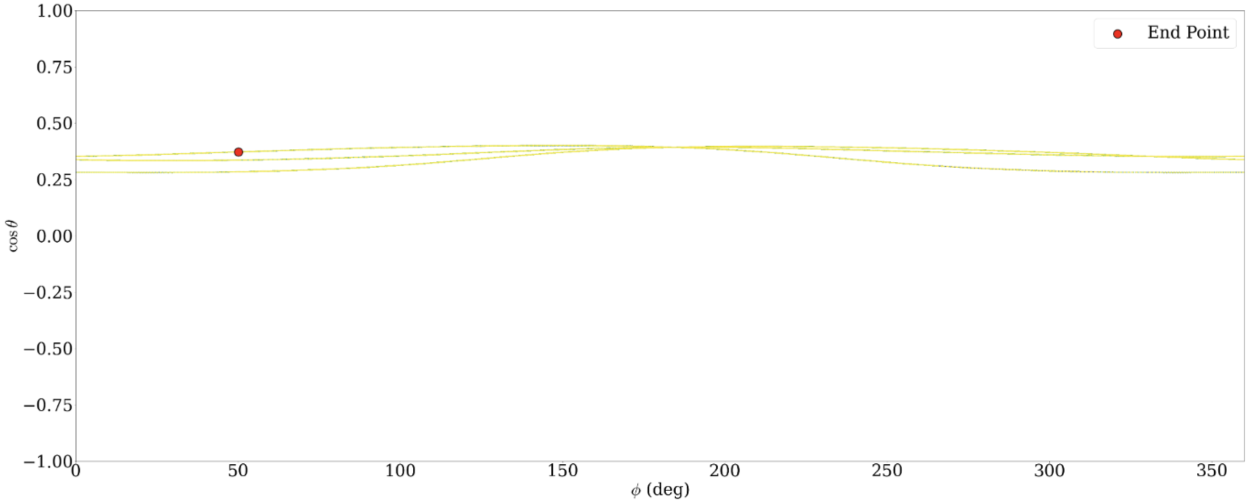
Non-Trivial Steady States

Beyond small I_2 ...









General Theory for Non-Trivial Steady States

We can look 2nd companion as a driver

- (Reminder) model baseline: Colombo's top
- 2nd companion as a (perturbative or not) driver
- Tidal effects dissipates energy/Hamiltonian

Over a geometric recurrence period,
we need a **energy/Hamiltonian balance!**

$$\oint_{\mathrm{circ}} \mathrm{d}t \left[\left(\frac{\mathrm{d}H}{\mathrm{d}t} \right) \Bigg|_{\mathrm{drive}} + \left(\frac{\mathrm{d}H}{\mathrm{d}t} \right) \Bigg|_{\mathrm{tide}} \right] = 0$$

How could this be always true?

Through non-linear RESONANCE!
Between driving frequency
and
libration/circulation frequency

$$m\Omega_{\mathrm{drive}} = n\omega_{\mathrm{libration/circulation}}$$

Perturbation Theory for Non-Trivial Steady States

$\epsilon \equiv \frac{I_2}{I_1} \ll 1$ as perturbation:

1:1 stable resonance condition:

$$\Omega_{\text{drive}} = \Delta g = \omega_{\text{libration/circulation}}$$

$$\oint_{\text{one circle}} \mathrm{d}t \left[\left(\frac{\mathrm{d}H_0}{\mathrm{d}t} \right)\Bigg|_{\text{drive}} + \left(\frac{\mathrm{d}H_0}{\mathrm{d}t} \right)\Bigg|_{\text{tide}} \right] = 0$$

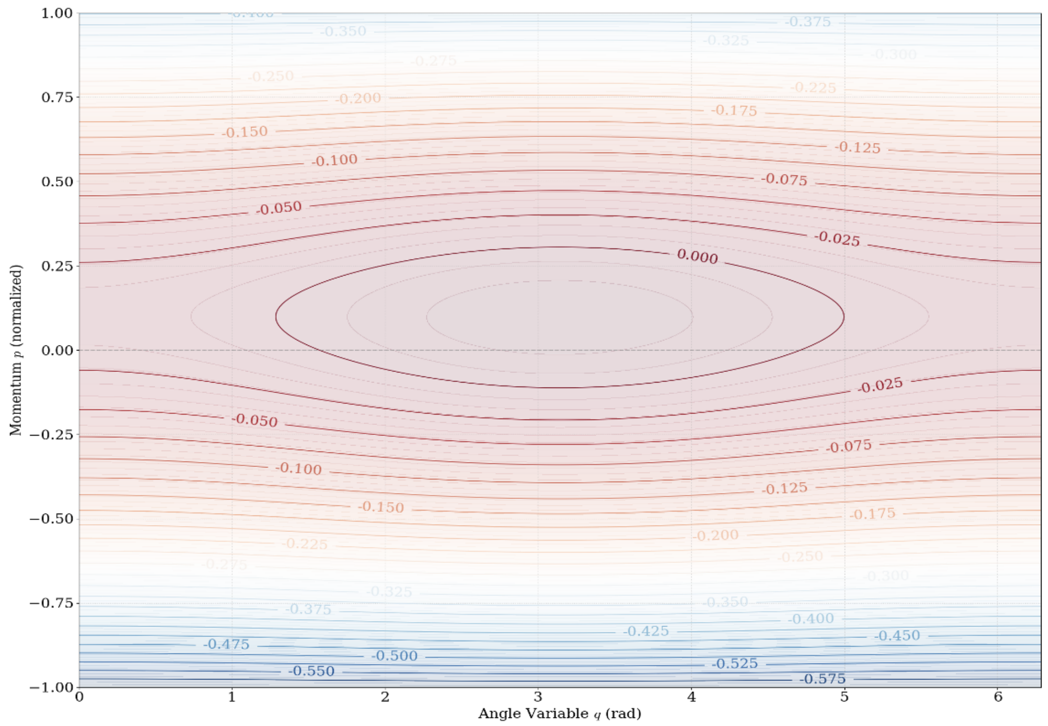
- Frequency condition defines the **resonant trajectory** in phase space (as a H_0 contour)
- Energy balance condition **verifies the existence**

$$H = H_0 + \epsilon H_1, \epsilon \ll 1$$ Hamiltonian P.T. formalism

$$H_0(p, q) = -\frac{1}{2}p^2 + g(p \cos I - \sin I \cos q \sqrt{1 - p^2})$$

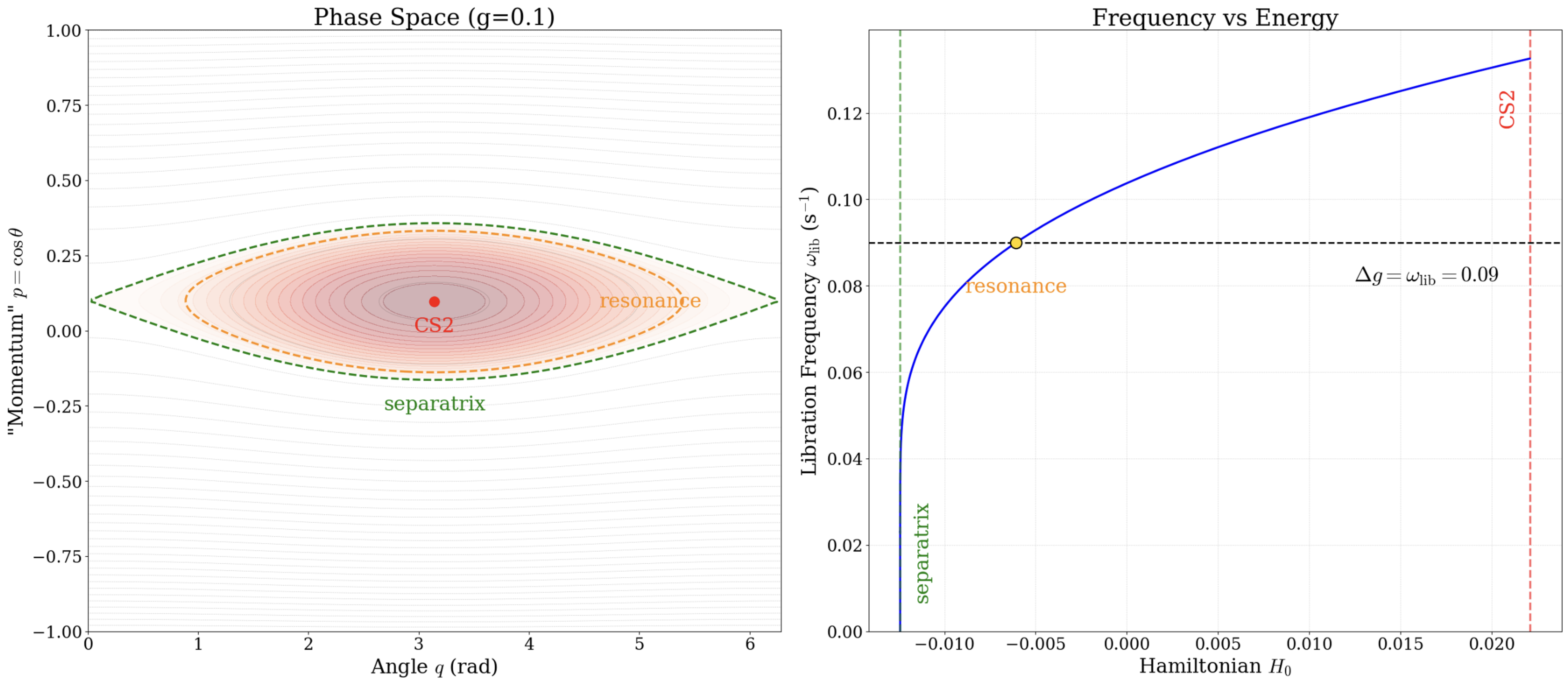
$$H_1(p, q, t) = -\Delta g \cos(\Delta g t)(p\sqrt{1 - I^2} - \sqrt{1 - p^2} I \cos q)$$

non-Hamiltonian dissipation $\dot{p}\Big|_{\text{tide}} = t_{\text{al}}^{-1}(1 - p^2)$



Perturbation Theory for Non-Trivial Steady States

1:1 stable resonance (libration) $I_1 = 10°, I_2 = 1°; g_1 = 0.1, g_2 = 0.01, \Delta g = 0.09$



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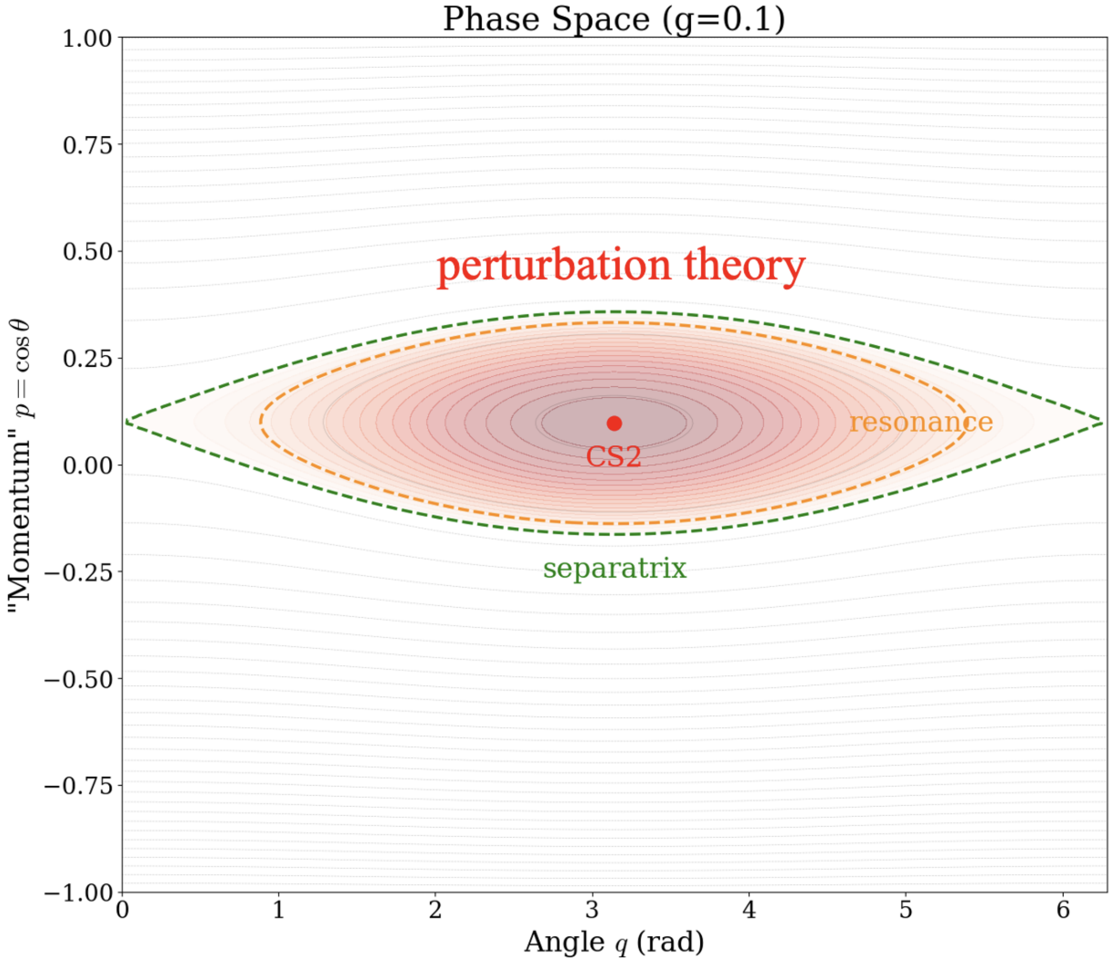

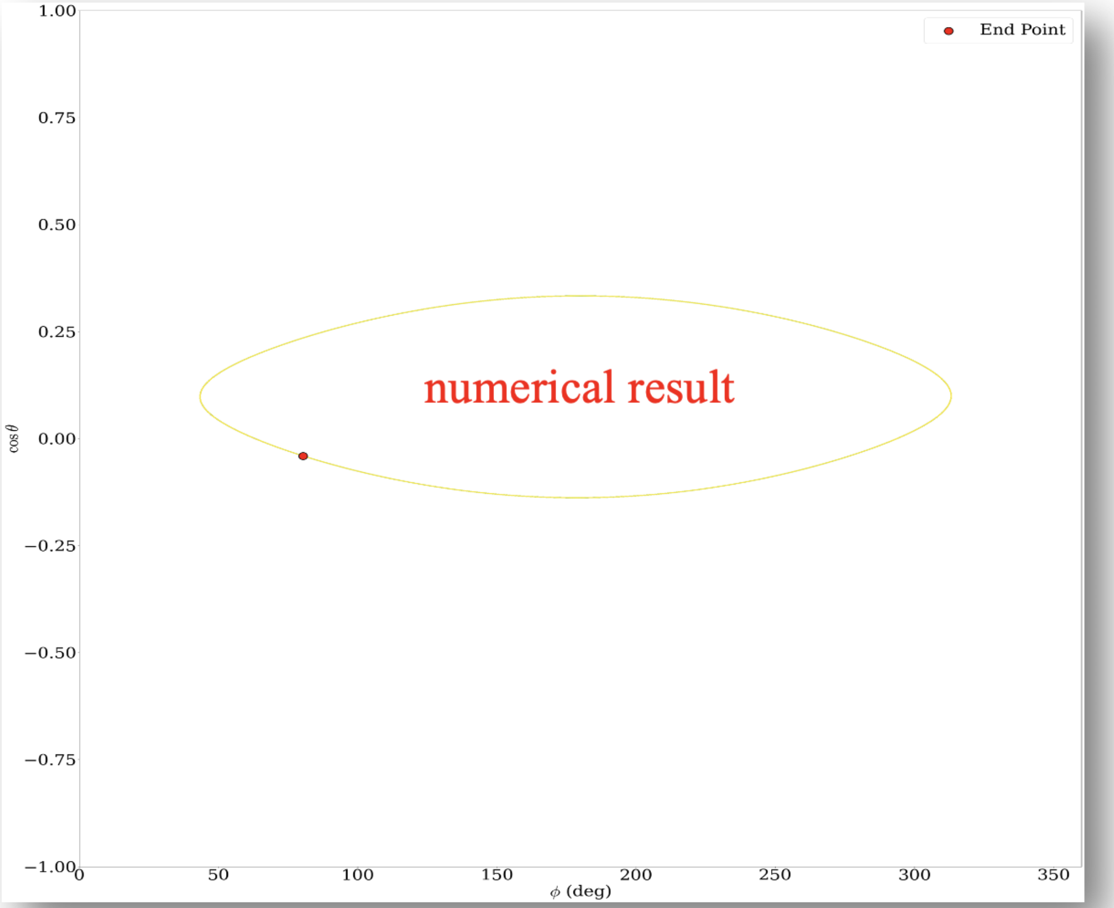

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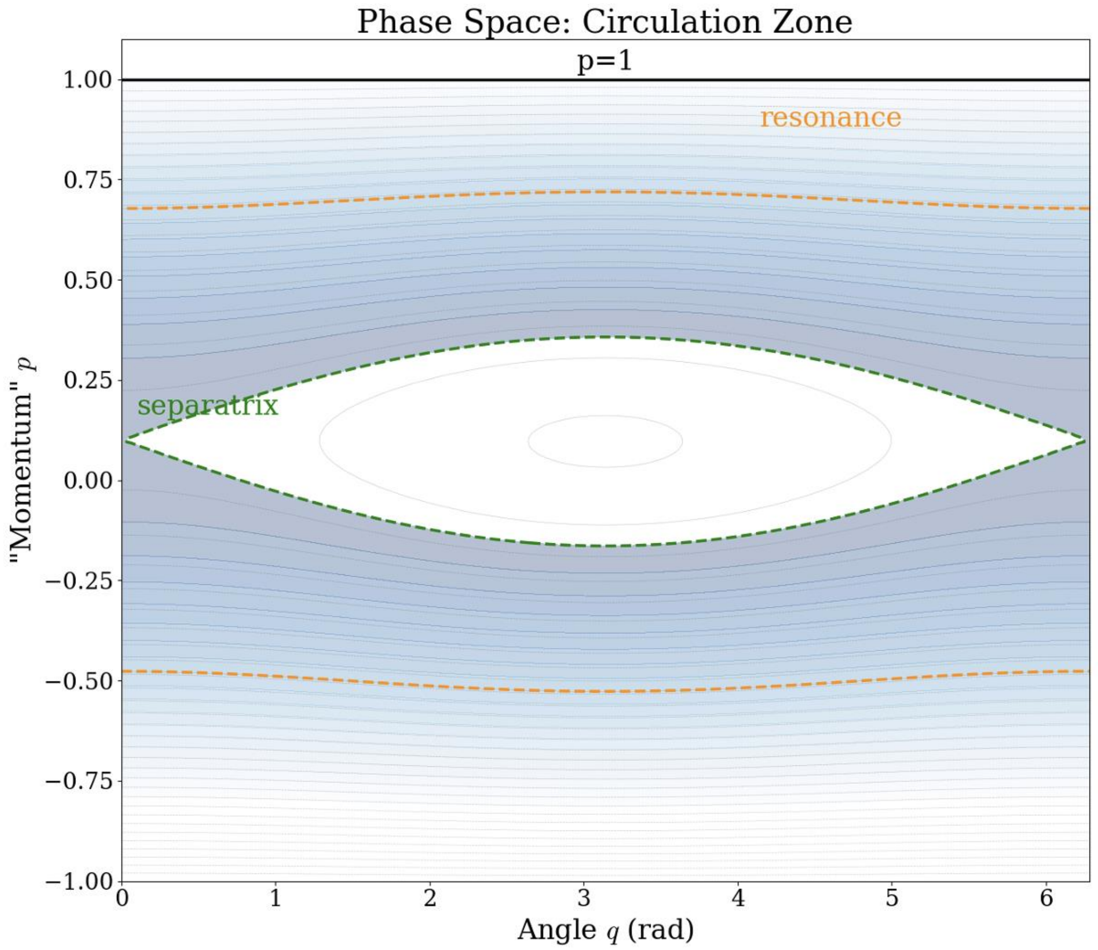

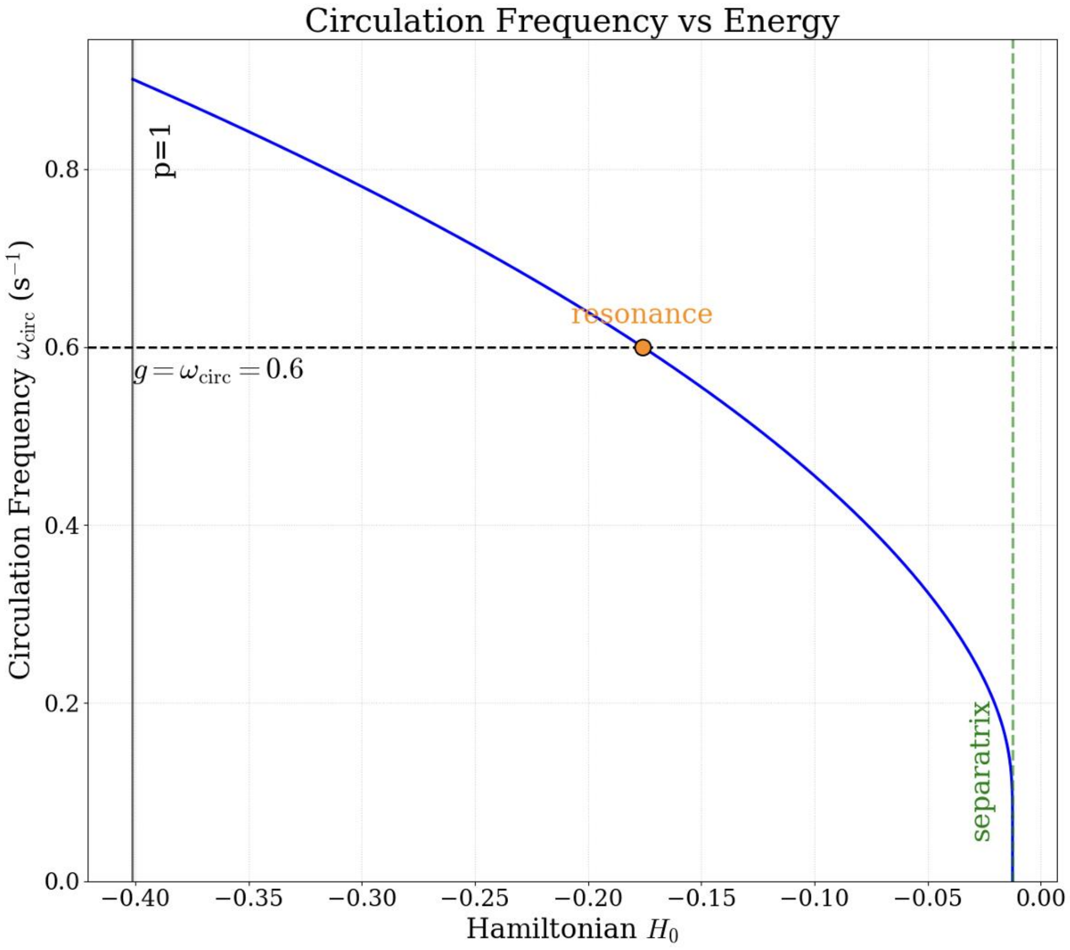

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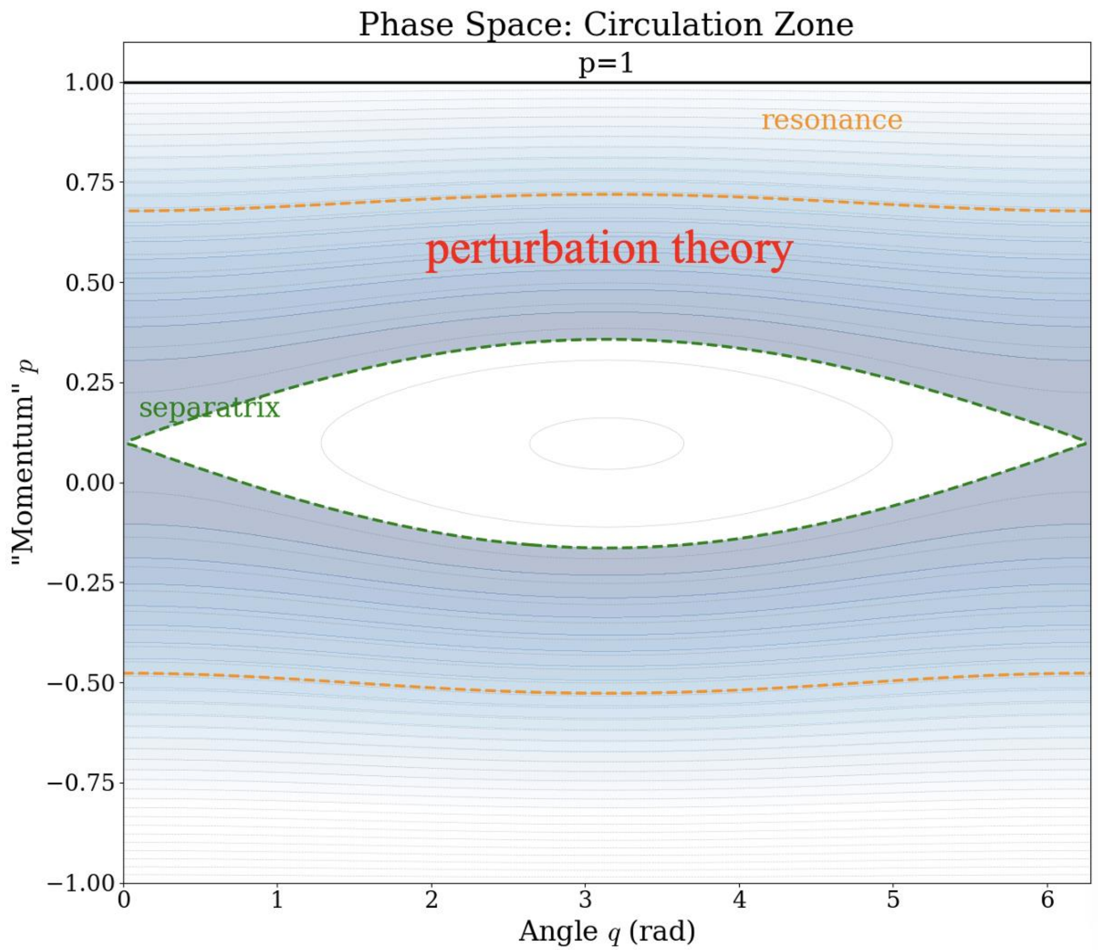
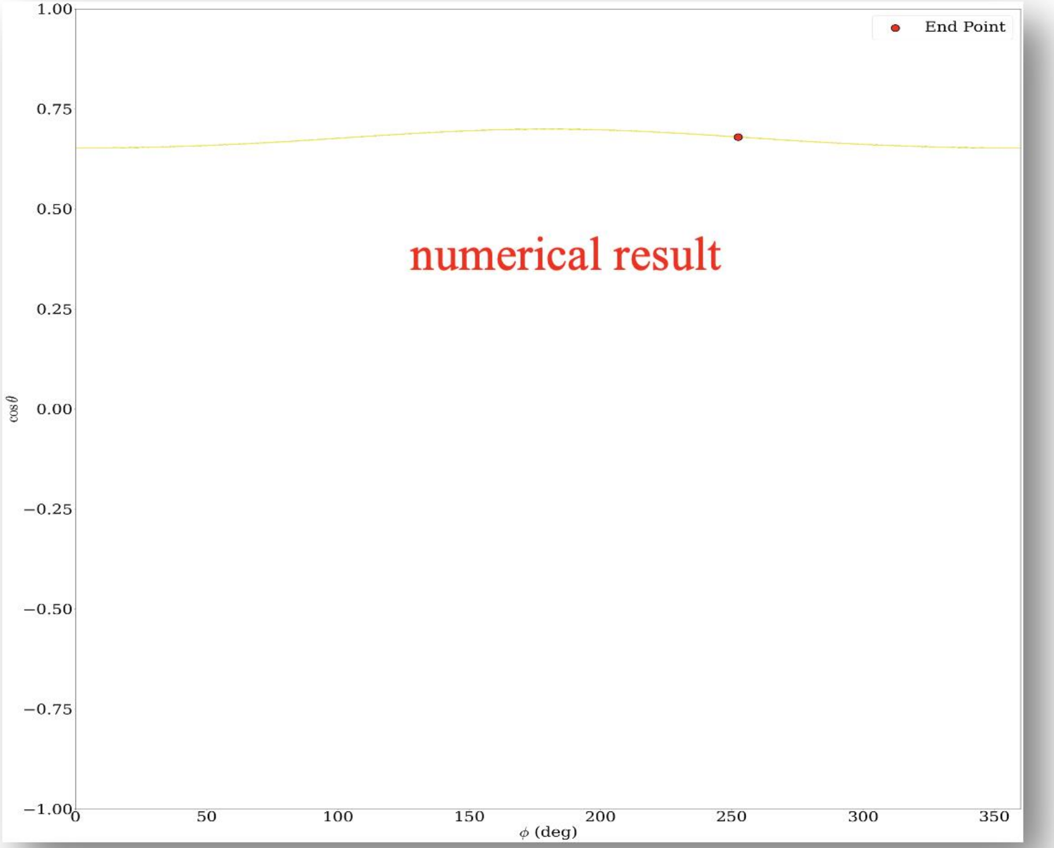
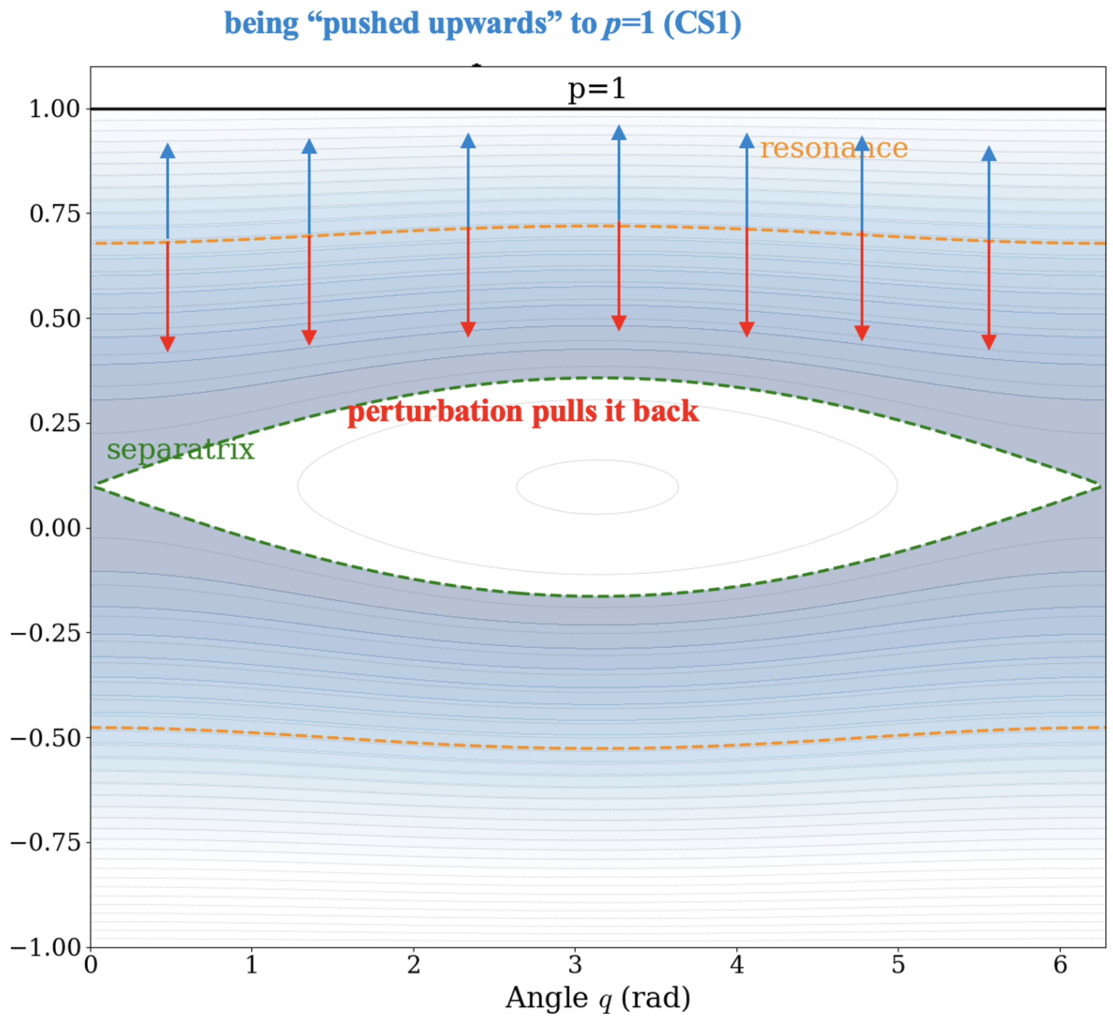
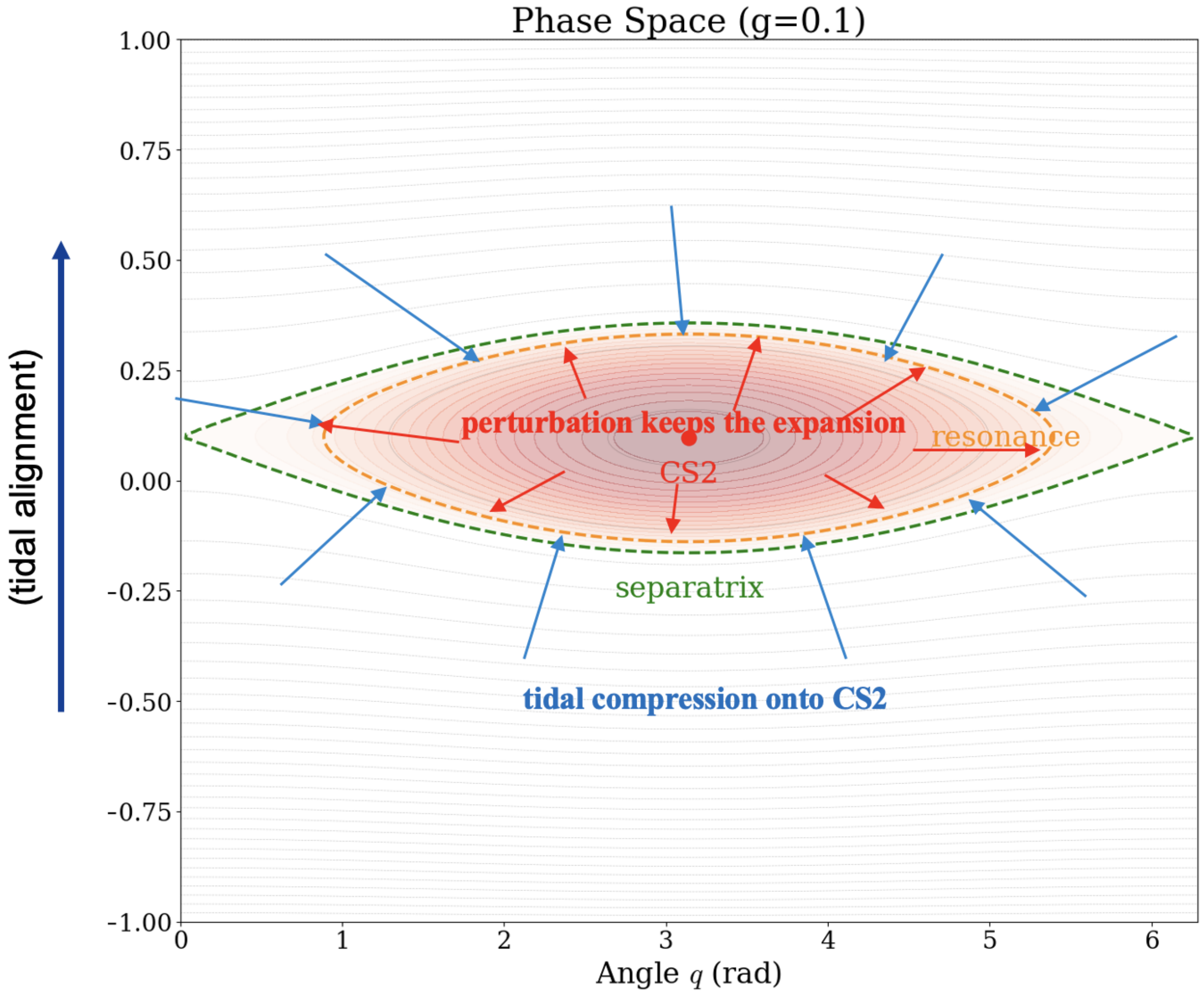




What happens here?

1:1 stable resonance





Further Questions...

- Where do these attractors come originate from model baseline?
  - CS4 / Separatrix

- How did our attractors reach the Hamiltonian balance?
  - Resonant mode-locking, a fixed $\Delta\varphi$ between driving & motion
  - Posting a constraint on t_{al}, ϵ

- What if our stable resonance conditions break up?
  - Non-linear attractors will collapse into the linear zone (around CS1/2)

- Beyond 1:1 resonance...

- Recovering conservative case and chaos...